JOURNAL OF AIRCRAFT
Vol. 46, No. 3, May–June 2009

Engineering Notes

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Flutter Equation as a Piecewise Quadratic Eigenvalue Problem

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DOI: 10.2514/1.40653

Introduction

FLUTTER is a dynamic aeroelastic stability problem which is usually analyzed by solving a nonlinear eigenvalue problem. Most existing solution methods are characterized by two distinct features, namely the interpolation of frequency–domain aerodynamic loads and an iteration procedure to solve the eigenvalue problem. The proposed approach exploits the properties of a C^1 continuous load interpolation scheme to solve the flutter eigenvalue problem without iteration.

Pitt [1] presented a noniterative scheme where the solutions of the linearized eigenvalue problem at fixed reduced frequencies are interpolated to other frequencies, which yields a computationally efficient procedure. However, that approach does not include means to determine the accuracy of the solutions obtained, and it requires a mode tracking procedure which can become fairly involved for complex cases with hundreds of modes. Bäck and Ringertz [2] eliminate the need for mode tracking and achieve a given accuracy, but their method requires iteration. Goodman [3] solves the flutter equation as a piecewise quadratic eigenvalue problem (QEP) as it is proposed here. Because of the type of interpolation used, the roots obtained thus are not continuous between segment boundaries. This problem is addressed with the interpolation scheme described below.

In the next section, the eigenvalue problem is presented, followed by a description of the interpolation method. Finally, a solution procedure based on this interpolation scheme is discussed.

Equations of Motion

Assuming small deformations and linear elastic materials, the equations of motion for the aeroelastic system can be written as a nonlinear eigenvalue problem according to

$$\left(\frac{u_\infty^2}{b^2}\boldsymbol{M}p^2 + \boldsymbol{K} - q_\infty \boldsymbol{Q}(p)\right)\hat{\boldsymbol{x}} = 0 \tag{1}$$

where $\boldsymbol{M}$ is the mass matrix and $\boldsymbol{K}$ the stiffness matrix obtained from a discretization of the structural system. The aerodynamic load matrix $\boldsymbol{Q}(p)$ depends on the eigenvalue p, which makes the problem nonlinear. The complex eigenvalue can be written as $p = g + ik$, where g is the damping and k is the reduced frequency.

To solve (1), the dependence of $\boldsymbol{Q}$ on p must be known. Methods for the computation of frequency–domain unsteady aerodynamic loads usually provide the load matrices $\boldsymbol{Q}$ for a discrete set of eigenvalues p. Often, a further approximation $\boldsymbol{Q}(p) \approx \boldsymbol{Q}(ik)$ is introduced because some methods are restricted to the computation of loads due to purely harmonic motion ($g = 0$).

With these restrictions, the direct solution of (1) can be performed by iterative methods. Typically, the eigenvalue problem is solved by introducing a parameter $\bar{p}$ so that $\boldsymbol{Q} = \boldsymbol{Q}(\bar{p})$, combined with a search or optimization procedure to find solutions of the resulting linear eigenvalue problem where $p = \bar{p}$, which, hence, are also solutions of the nonlinear problem (1). For an efficient solution procedure, some kind of interpolation is usually employed to avoid the repeated direct evaluation of $\boldsymbol{Q}$. The idea presented in this note is to exploit this interpolation of $\boldsymbol{Q}(p)$ to reformulate the nonlinear Eq. (1).

Piecewise Quadratic Interpolation

Given a set of aerodynamic load matrices $\boldsymbol{Q}(k_i)$ which can be computed by all methods for frequency-domain aerodynamics, a piecewise quadratic interpolant is constructed along the imaginary axis. If the reduced frequencies at which the load matrices are available are designated k_i, with $i \in [1, n]$ then the $n-1$ breakpoints b_i for the piecewise interpolation are chosen as

$$b_1 = k_1 \tag{2}$$

$$b_i = (k_i + k_{i+1})/2 \quad \text{for } i \in [2, n-2] \tag{3}$$

$$b_{n-1} = k_n \tag{4}$$

For each segment j so that $k \in [b_j, b_{j+1}]$, three conditions are imposed, namely that the value of the previous segment $j-1$ is matched at b_j, that the slope of the following segment is matched at b_{j+1} and that the discrete value of $\boldsymbol{Q}(k_{j+1})$ is interpolated exactly. For the first and last segment, the conditions are modified to interpolate the first and last $\boldsymbol{Q}$ instead.

With three conditions for each segment, a piecewise quadratic interpolant

$$\boldsymbol{Q}\,(p) = \boldsymbol{Q}_j^0 + \boldsymbol{Q}_j^1 p + \boldsymbol{Q}_j^2 p^2 \quad \mathrm{Im}(p) \in [b_j, b_{j+1}] \tag{5}$$

can be computed, where $\boldsymbol{Q}_j^i$ are the polynomial interpolation coefficients for segment j and the superscript i is to be understood as an index, not an exponent.

Properties of the Interpolation Scheme

Even if the loads are only computed on the imaginary axis, the quadratic interpolation (5) provides a second-order approximation of the variation with g. This is a result of the formulation of (5) over the full complex plane, which fulfils

$$\frac{\partial \boldsymbol{Q}}{\partial g} = -ig\frac{\partial \boldsymbol{Q}}{\partial k} \quad \text{and} \quad \frac{\partial^2 \boldsymbol{Q}}{\partial g^2} = -\frac{\partial^2 \boldsymbol{Q}}{\partial k^2} \tag{6}$$

Received 28 August 2008; accepted for publication 3 February 2009.

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the Cauchy–Riemann equations for analytical functions. These must be valid if it is assumed that $\boldsymbol{Q}$ is differentiable on the interior of the half-plane $k \geq 0$. To the author's knowledge, there is neither theoretical nor experimental evidence which suggests that this should not be the case. When using (5), the Cauchy–Riemann conditions are fulfilled on the whole imaginary axis and on the interior of each quadratic segment.

For flutter analysis, solutions with $g = 0$ are of primary interest, and for these a smooth interpolation is achieved. At large (positive or negative) damping values, however, the interpolation yields discontinuous values across the segment boundaries, which can result in jumps in the eigenvalue $\mathrm{Re}(p)$ whenever $\mathrm{Im}(p)$ crosses any of the breakpoints b_j, and $|g|$ is large. Figure 1 shows these discontinuities for a single component of the load matrices computed for the example presented in the next section. The empty circles represent discrete values computed by the aerodynamic solver, and the solid black line is the piecewise parabolic interpolant along the imaginary axis. Cross and triangle symbols show the result of evaluating this interpolant at different damping values.

The magnitude of the discontinuities depend on g and on the size of the interpolation segment. Near the imaginary axis, the differences across interpolation segment boundaries are very small, but grow to considerable magnitude for sufficiently large g. These jumps can be reduced drastically by using more interpolation segments. The necessary additional interpolation points can be obtained by evaluating the interpolant at more points on the imaginary axis where it is smooth. There is, hence, no need for additional calls to the underlying aerodynamic solver. The solid gray lines in Fig. 1 show the result of evaluating this refined interpolant with 120 instead of the original 13 segments. At this resolution, the discontinuities between segments are extremely small even at large g.

Quadratic Eigenvalue Problem

Once a piecewise quadratic interpolation for $\boldsymbol{Q}$ is available, Eq. (1) can be posed as a standard quadratic eigenvalue problem [4] for each segment j as in

$$\left[\left(\frac{u_\infty^2}{b^2}\boldsymbol{M} - q_\infty \boldsymbol{Q}_j^2\right)p^2 - q_\infty \boldsymbol{Q}_j^1 p + \boldsymbol{K} - q_\infty \boldsymbol{Q}_j^0\right]\hat{\boldsymbol{x}} = 0 \qquad (7)$$

A standard solution procedure for this type of problem is the reformulation into a linear eigenvalue problem with twice the number of variables. Solving this augmented problem yields exactly twice as many eigenvalues as the dimension of the original system, which are not complex conjugate in general. Since the interpolant (5) used in this form is only valid within the segment j, only solutions with $\mathrm{Im}(p) \in [b_j, b_{j+1}]$ are kept. Thus, the solution of (1) reduces to $n - 1$ linear (but larger) eigenvalue problems, resulting in all solutions p for which $\mathrm{Im}(p) \in [k_1, k_n]$.

The solution procedure in the proposed quadratic eigenvalue form is relatively simple compared with modern, robust implementations of the p-k method [2] or the g-method [5]. Furthermore, it does not require any iterative procedures which may fail to converge or may need user supplied tolerance criteria for termination. Finally, the solution of the small quadratic eigenvalue problem on each interpolation segment is independent of all other segments, and thus inherently parallel.

In Goodman's algorithm [3], eigenvalues are blended at segment boundaries to smooth discontinuities introduced by the interpolation scheme. As shown above, the interpolant proposed here is C^1 continuous on the stability boundary and provides an inherent facility to reduce the possible discontinuities at large damping values to arbitrarily small levels. Therefore, neither mode tracking nor blending of roots is required.

Example Application

The flutter problem is solved for the ASK 21 sailplane for which a validated NASTRAN finite-element model is available [6,7]. Generalized aerodynamic load matrices $\boldsymbol{Q}$ were computed with a boundary-element method for incompressible potential flow [8]. The discretization used is an unstructured triangular surface mesh with 25 000 elements. A standard modal-subspace procedure is employed for the solution of (1) using the first 26 structural eigenmodes with frequencies up to 38 Hz. The flutter problem is solved both using the modified p-k method of Bäck and Ringertz [2] and the proposed piecewise QEP.

In Fig. 2, the development of the damping g of six aeroelastic modes is shown as a function of airspeed. The modes were selected by the amount of variation in damping over velocity.

As expected, the computed eigenvalues p match exactly at the flutter boundary where $g = 0$. Differences start to appear for damping values of $|g| > 0.05$, although the difference in frequency (not shown) and in damping remain below 10%. Larger deviations are only observed for aeroelastic modes with large damping, that is, for $g < -0.3$. Such strongly damped solutions, where g approaches or exceeds $-k$, should, however, be considered as dubious regardless of the method of solution. The motion predicted by these eigenvalues is (almost) critically damped, that is, not truly oscillatory, while the aerodynamic loads used to arrive at the solution are obtained for pure harmonic oscillation. To correctly predict strongly damped aeroelastic motion, aerodynamic loads for this type of motion must be computed directly and not just extrapolated from the undamped case.

Even though only 29 interpolation segments were used in this case, this particular solution does not exhibit any relevant discontinuities. Closer examination shows that small jumps do in fact show up for eigenmodes with very large damping values. Given the limited validity of such solutions, this disadvantage is not considered severe.

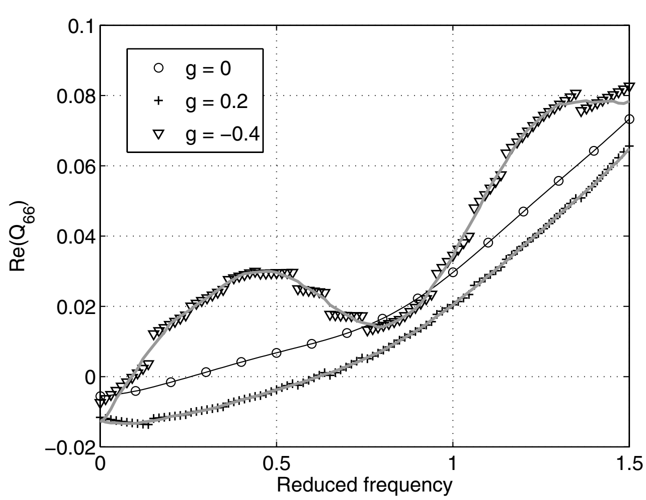

Fig. 1 Interpolation evaluated at different g.

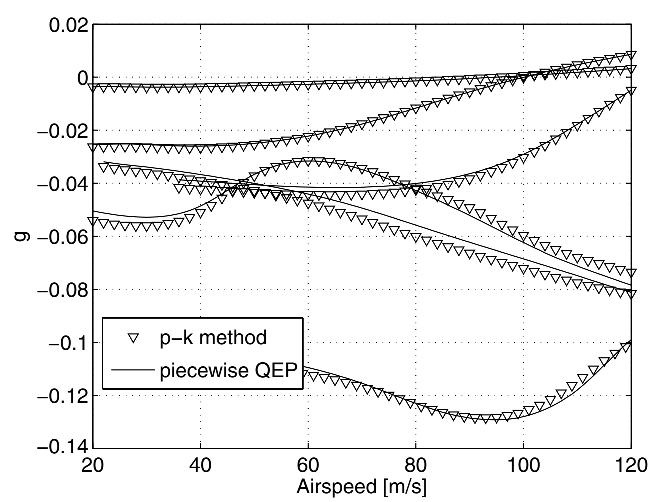

Fig. 2 Comparison of flutter solutions.

Conclusions

The advantage of the proposed method for the solution of the flutter problem is its robustness and the inherent inclusion of the effect of damping on the aerodynamic loads near the imaginary axis. This is achieved while still using aerodynamic data computed for purely harmonic motion, which is a common limitation in methods for the computation of unsteady aerodynamic loads. An application example shows that the flutter speed is predicted at exactly the same airspeed and frequency as with the *p*-*k* method, while the predicted damping of subcritical aeroelastic modes differs moderately.

A possible application of the method may be the improved prediction of expected damping in flutter flight testing. In this case, the flutter boundary is approached from subcritical airspeeds where $g < 0$. Independent of the means used to arrive at the aerodynamic loads, the presented flutter solution approach better accounts for damping than the *p*-*k* method and should thus provide better correlation in the case of subcritical motion.

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